

3.4.11)

a) $\frac{dx}{dt} = rx - \sin(x)$ for $r = 0 \Rightarrow$ to find fixed points, $\sin(x^*) = 0 \Rightarrow x^* = n\pi$ where $n \in \mathbb{Z}$

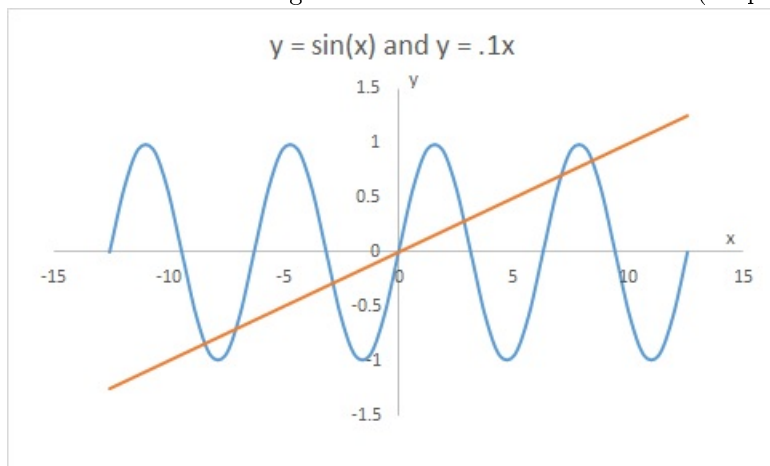
Stable when n is odd and unstable when n is even [1 point]

b) $\frac{dx}{dt} = rx - \sin(x)$ for $r > 1 \Rightarrow$ to find fixed points, $rx - \sin(x^*) = 0$

Let $f(x) = rx - \sin(x) \Rightarrow f'(x) = r - \cos(x) \Rightarrow$ since $r > 1 \Rightarrow f'(x) > 0 \Rightarrow f(x)$ is strictly increasing and $f(x) = 0$ only when $x = 0 \Rightarrow$ Only one fixed point at $x^* = 0$, unstable as $f'(0) > 0$

Can also show this graphically (plot both rx and $\sin x$ and show they only intersect at $x = 0$) [1 point]

c) As r decreases from infinity a subcritical pitchfork bifurcation occurs at $r = 1$ as two fixed points are created. As r decreases from 1 to 0 the line rx intersects more and more points on the sine function with each intersection creating a set of saddle node bifurcations (for positive and negative x)



[1 point for description of both types of bifurcations]

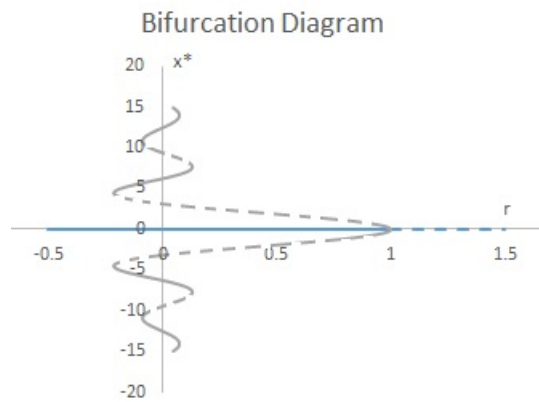
d) For saddle node bifurcations $f(x) = 0$ and $f'(x) = 0 \Rightarrow r = \frac{\sin(x)}{x}$ and $r = \cos(x)$.

Since $r \ll 1 \Rightarrow \cos(x) \ll 1$ which means $\cos(x) \approx 0 \Rightarrow x^* \approx \frac{\pi}{2} + 2n\pi$ (can only approach 0 from positive side as $r > 0$) [1 point for x^*]

Inserting this into $r = \frac{\sin(x)}{x}$ we get $r_c = (-1)^n (\frac{\pi}{2} + 2n\pi)^{-1} \Rightarrow$ since $r > 0$, $r_c \approx (\frac{\pi}{2} + 2n\pi)^{-1}$ [1 point for r_c]

e) As r decreases from 0 to $-\infty$ the saddle node bifurcations begin to disappear. [1 point]

f)

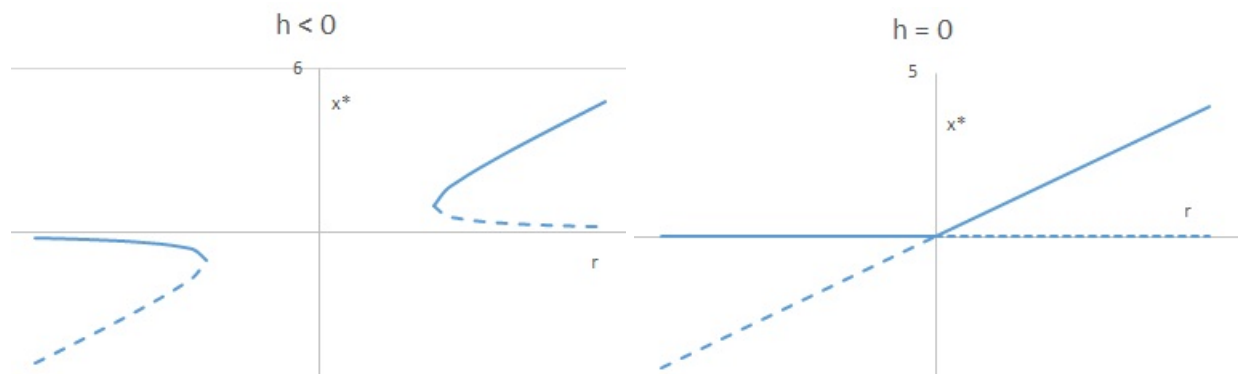


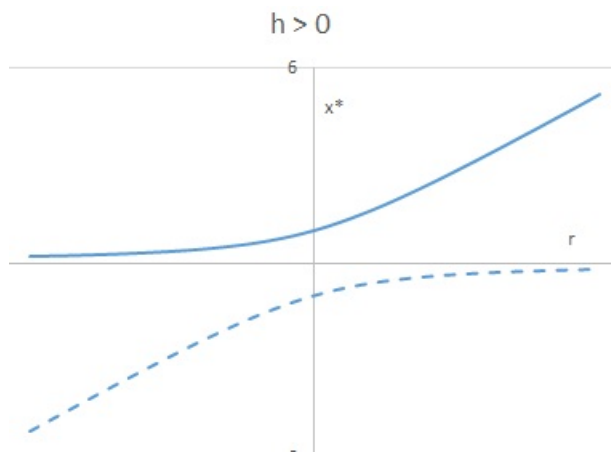
[1 point of correct graph and 1 point for correct stability]

Total 8 points

3..6.2)

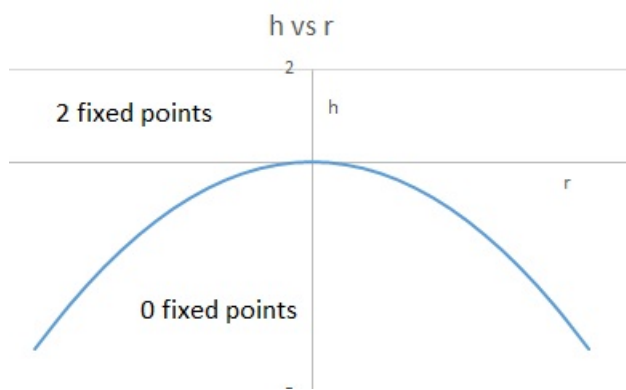
a)





[3 points, 1 for each plot]

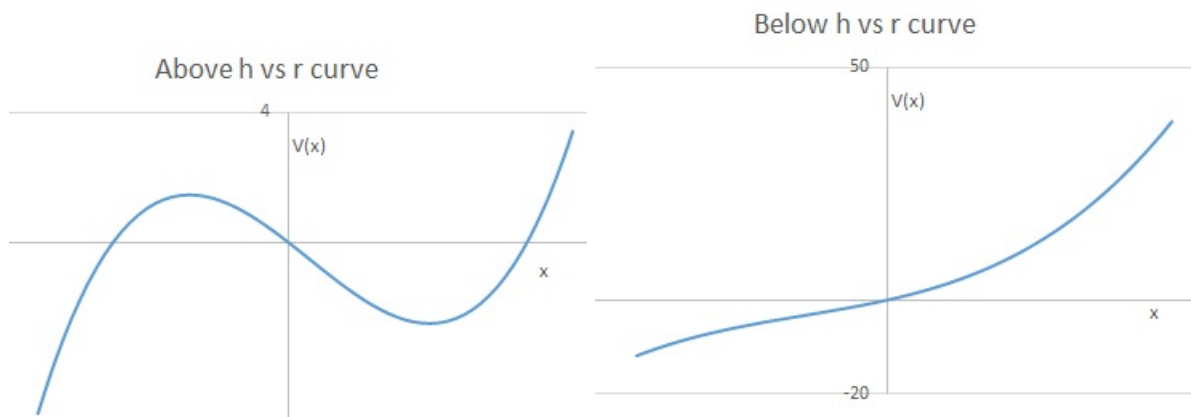
b) $\frac{dx}{dt} = 0 = h = rx - x^2 \Rightarrow x^* = \frac{r}{2} \pm \frac{\sqrt{r^2 + 4h}}{2}$ From a) it's clear that a bifurcation occurs when there's only one fixed point $\Rightarrow$ bifurcation occurs when $\sqrt{r^2 + 4h} = 0 \Rightarrow h = -\frac{r^2}{4}$ [1 point]



[1 point for plot]

c)

$\frac{dx}{dt} = -\frac{dV}{dx} \Rightarrow V(x) = -hx - r\frac{x^2}{2} + \frac{x^3}{3}$ Two regions: Above the h vs r curve and below it



, [2 points, 1 for each plot]

Total 7 points